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The Decline of Cayley's Invariant Theory (1863–1895)

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From the 1860s the German symbolic approach to invariant theory was in ascendancy. This article discusses the work of Arthur Cayley (1821–1895) and his reaction to this new line of enquiry. The symbolic method is outlined and compared with Cayley's viewpoint in which the calculation and exhibition of invariants and covariants were of primary importance. Cayley's Law and Gordan's finiteness theorem, two principal results in the theory, are discussed. Also covered is J. J. Sylvester's Fundamental Postulate, which both reveals the character of the English empirical approach to invariant theory and illustrates its inherent weakness. The article examines the background to Cayley's final three memoirs on quantics, his last work in invariant theory, and it makes use of correspondence with his friend Sylvester. © 1988 Academic Press, Inc.

In diesem Aufsatz werden die von dem Mathematiker Arthur Cayley (1821–1895) zwischen 1863 und 1895 geleisteten Beiträge zur Invariantentheorie diskutiert. Da die deutsche symbolische Methode sich während dieser Zeit als zunehmend einflussreich erwies, vergleicht der Autor diese Methode mit Cayleys Ansicht, nach der die Berechnung und Darstellung von Invarianten von grosser Bedeutung ist. Nach einer ausführlichen Erörterung von Cayleys Gesetz und Gordans Endlichkeitssatz, zwei wichtigen Ergebnissen der Invariantentheorie, wird darauf hingewiesen, dass das von Sylvester entwickelte "Fundamentale Postulat" eine der englischen Annäherungsmethode eigenartige Schwäche offenbart. Bei der Behandlung der Beiträge Cayleys zur Invariantentheorie bezieht sich der Autor auf den Briefwechsel zwischen Cayley und seinem Freund J. J. Sylvester. © 1988 Academic Press, Inc.

A partir des années 1860, l'approche symbolique des allemands en théorie des invariants tend à prédominer. Cet article porte sur l'oeuvre d'Arthur Cayley (1821–1895) et sur ses réactions à cette nouvelle orientation. Nous présentons sommairement la méthode symbolique et la comparons à l'approche de Cayley dans laquelle le calcul et la mise en évidence des invariants et des covariants furent de première importance. Nous analysons deux résultats fondamentaux de la théorie soient la loi de Cayley et le théorème des nombres finis de Gordon. Par le biais du postulat fondamental de J. J. Sylvester, il nous est possible à la fois de mettre en évidence le caractère empirique de l'approche anglaise en théorie des invariants et d'illustrer la faiblesse inhérente à celle-ci. La correspondance avec son ami Sylvester nous permet un examen des antécédents aux trois derniers mémoires de Cayley sur les quantiques et à son dernier travail en théorie des invariants. © 1988 Academic Press, Inc.

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INTRODUCTION

By the beginning of the 1860s Arthur Cayley could look back on a 20-year period of pioneering work in invariant theory. He had laid the groundwork for the

two great methods of 19th-century invariant theory and begun the important task of classifying invariants and covariants. Seven of his famous 10 memoirs on quantics had been completed and he had decided that invariant theory should be based on partial differential equations instead of his earlier method of hyperdeterminant derivation. This pathbreaking work had been carried out when Cayley was first at Cambridge and while he was associated with Lincoln's Inn as a barrister. His work in invariant theory during this period (1841–1862) has been described (see my [1986]).

In 1863 Cayley was elected to the newly established Sadleirian chair of pure mathematics at Cambridge University. Although invariant theory continued to be one of his central interests, he published spasmodically in the subject during the 1860s and 1870s. During this period, using a more streamlined approach than the one adopted by Cayley, the German mathematicians Siegfried Aronhold (1819–1884), Alfred Clebsch (1833–1872), and Paul Gordan (1837–1912) made greater progress.

While Cayley appreciated the significance and power of the German symbolic method, it is notable that he did not adopt it for his own work. The method provided a calculus for the calculation of invariants and covariants but it was perhaps too abstract in character for the tastes of the English school. For Cayley, the success of the symbolic method lay in the establishment of theoretical results, including Gordan's Theorem. But in addition to theoretical results Cayley saw as one of the central objectives of the subject the computation of invariants and covariants written in terms of homogeneous coordinates. These lengthy algebraic forms, many of which contained terms numbered in the thousands, I have chosen to call *Cartesian expressions* (see glossary in Appendix) [1]. Cayley's differential equation approach facilitated the computation of invariants and covariants as Cartesian expressions while the German method appeared to offer no improvement in this direction.

An effect of Cayley's line of enquiry was his failure to develop a sound theoretical calculus, and this hindered progress toward a mature theory. Instead, Cayley and his friend J. J. Sylvester (1814–1897) adopted a pragmatic approach to invariant theory and some of their practices were based on little more than unproven hypotheses. One particular example, Sylvester's Fundamental Postulate, was an assumption extrapolated from established results relating to binary forms of low order. Although tempting to believe, the postulate was eventually found to be false. Another illustration of their pragmatic spirit was *Cayley's Law* (see glossary in Appendix). This law gave a formula for enumerating linearly independent covariants of a binary form. Cayley's mathematical argument, however, contained a gap; he took for granted that a certain set of linear equations was independent, perhaps as a result of inductive evidence. Cayley's intuition was quite correct though the assumption required a nontrivial argument (supplied by Sylvester 20 years later).

In my [1986] I argued that the period (1841–1862) saw the "rise" of Cayley's invariant theory. In this earlier period Cayley's approach led to lasting results

(e.g., Cayley's Law) that later generations continued to derive by methods similar in spirit to Cayley's. During the later period (1863–1895) his objectives and approach did not fare well compared to the triumphs of the German symbolic method.

THE ENGLISH APPROACH

The English approach to invariant theory was based on the idea that an invariant could be regarded as a solution to a set of partial differential equations. Here we mean the purely algebraic properties associated with partial differential equations rather than analytic properties. To use Sylvester's terminology, an invariant was an algebraic form *annihilated* by differential operators. Thus an invariant I of the binary form

$$a_0x^n + a_1 \binom{n}{1} x^{n-1}y + a_2 \binom{n}{2} x^{n-2}y^2 + \dots + a_n y^n$$

[2], written as a Cartesian expression, is annihilated by the differential operators \square , \square (see my [1986, 245–247]). Since this view facilitated the calculation of invariants and covariants, it suited the English approach to invariant theory and became its hallmark. Sylvester demonstrated his commitment to differential operators when he grandly described them as the “subtlest of all instruments for putting Nature and Reason to the question” [1877; SP3, 86].

Mid-century mathematicians were well versed in the calculus of operations and it is not surprising to see their application of it to invariant theory. One of the well-tried operator identities which Cayley extensively used—and one which reminds us of Cayley's mathematical perspective—was

$$P \cdot Q = PQ + P(Q).$$

In this, $P \cdot Q$ is the ordinary composition of operators with Q operating first then followed by P . The term PQ is the product of the operators multiplied together as ordinary commutative algebraic quantities and $P(Q)$ is the result of applying P to Q where Q is considered as an operand. In the case $P = a_1(\partial/\partial a_2)$ and $Q = a_2(\partial/\partial a_1)$, for example,

$$P \cdot Q = a_1 a_2 \frac{\partial}{\partial a_1} \frac{\partial}{\partial a_2} + a_1 \frac{\partial}{\partial a_1}.$$

According to [Koppelman 1971, 193], this identity had long been used by British mathematicians of the 19th century. The idea of splitting the composition $P \cdot Q$ into a commutative part PQ and a noncommutative part $P(Q)$ had proved its usefulness in Cayley's work, and he continued to use it until the last [1893b; CP13, 400].

THE SYMBOLIC METHOD

The German symbolic method was based on a succinct notation for algebraic forms. In the symbolic notation the binary form of order n was written as $(\alpha_1 x_1 + \alpha_2 x_2)^n$ and further abbreviated to α_x^n . By comparing the formal binomial expansion

of this with Cayley's Cartesian expression for the binary form, we see that the symbolic product $\alpha_1^{n-i}\alpha_2^i$ represents the i th coefficient a_i . In the German symbolic notation the symbols α_1, α_2 have meaning only when combined together to form symbols of degree n . This representation of algebraic forms was referred to by the English as an *umbral* notation and although Cayley had lightly touched on a similar notation earlier, he failed to develop the idea [Cayley 1854b]. He also used the notations $(* \text{ } \text{ } x, y)^n$ and $(* \text{ } \text{ } x, y)^n$ quite freely as shorthand but did not think of them as a substitute for the extensive Cartesian expression itself.

As part of the symbolic method, the German mathematicians defined a symbolic determinant

$$(\alpha\beta) = \alpha_1\beta_2 - \alpha_2\beta_1$$

and this was used to represent invariants and covariants. In the case of the quadratic form,

$$ax^2 + 2bxy + cy^2,$$

for example, representing ac by $\alpha_1^2\beta_2^2$ (or equivalently by $\alpha_2^2\beta_1^2$) and b^2 by $\alpha_1\alpha_2\beta_1\beta_2$, the symbolic determinant is

$$(\alpha\beta)^2 = \alpha_1^2\beta_2^2 - 2\alpha_1\alpha_2\beta_1\beta_2 + \alpha_2^2\beta_1^2,$$

showing that $(\alpha\beta)^2$ can be used to represent the invariant of the quadratic form which Cayley preferred to write as $ac - b^2$. According to Meyer, the author of the authoritative *Bericht* on invariant theory, the theorem that any invariant could be written as a product of symbolic determinants signaled the superiority of the symbolic method [Meyer 1890, 188n]. (Symbolic expressions for *covariants* involved additional factors of the kind α_x^k .)

Coupled with their notation, the German mathematicians defined a binary operation on algebraic forms. This, the *transvection operation* (see glossary in Appendix) or derivative, gave a special power to the German method. A formal calculus was built on the basis of this operation, and its use enabled invariants and covariants to be generated at will. The result of applying the transvection operator, the k th transvectant, was denoted by $(\alpha_x^n, \beta_x^m)^k$. The first transvectant, for example, is the *Jacobian* covariant of α_x^n and β_x^m . In addition to the facility it provided for the calculation of invariants and covariants, the transvection operation possessed a theoretical importance. Indeed, Gordan's success in proving his finiteness theorem lay in the establishment of a key lemma in which it was crucially involved. In reviewing this work Cayley emphasized the importance of Gordan's lemma which stated that invariants and covariants of degree n could be obtained by transvection from the parent quantic and an invariant or covariant of degree $(n - 1)$ (illustrated for the *binary cubic* (see glossary in Appendix)).

But Cayley did express reservations and his general attitude toward the symbolic notation has some parallel with his view of the succinct notation of quaternions. Cayley compared the latter notation—which could be used to advantage in analytical geometry and which was a forerunner of the vector notation—“to a pocket-map—a capital thing to put in one's pocket, but which must be unfolded:

the formula, to be understood, must be translated into co-ordinates'' [Cayley 1895, 272]. In addition, there did not appear to be an easy transition from the symbolic notation of a covariant to the equivalent Cartesian expression.

It is important to emphasize that both Cayley and Sylvester studied the German method and in the ninth memoir Cayley presented a sketch of it for English readers. This synopsis showed the connection between the German method and Cayley's own youthful discovery, the hyperdeterminant derivation method. Moreover Cayley stressed the distinction between the two: he believed the hyperdeterminant derivative was a wider concept than the transvection operation. His hyperdeterminant derivative method was not confined to a single transvection operation, but allowed the composition of hyperdeterminant derivatives to act on any number of binary forms. Cayley wrote, for instance,

$$\overline{12^\alpha 13^\beta 23^\gamma} u_1 u_2 u_3$$

for binary forms u_1 , u_2 , and u_3 [1889; CP1, 585]. But the strong connection between the two methods led Sylvester to exclaim (in a letter to William Spottiswoode): "The piratical Germans, Clebsch and Gordan who have so unscrupulously done their best to rob us English of all the credit belonging to the discoveries made in the New Algebra will now suffer it is hoped the due Nemesis of their misdeeds" (2) [3]. However, as Cayley clearly acknowledged, Gordan successfully used the transvection operation to solve an outstanding problem of the day.

Despite its success, the symbolic method was not widely accepted and, according to W. F. Osgood, it had not met with acceptance in the English language journals by as late as the 1890s [Osgood 1892, 251]. The natural time lag in assimilating new ideas was a contributory factor as was the adherence of the English school to their own algebraic methods.

COMPUTATIONAL COMPLEXITY

While the German mathematicians based their work on an abstract calculus, they did not neglect calculation. As is well known Gordan successfully calculated the 23 irreducible invariants and covariants for the binary quintic and 26 for the binary form of order six [Gordan 1868]. The results captured Cayley's interest and they came at a time when he was failing to make discernible progress in his own work in invariant theory. In fact, 6 years passed between the appearance of the seventh and the eighth memoirs on quantics. The eighth memoir, published in 1867, began with the rather lifeless statement concerning covariants of the binary quintic form: "it was interesting to proceed one step further, viz. to the covariants of the degree 6" [Cayley 1867]. Moreover, Cayley must also have doubted the wisdom of pursuing the calculation of the irreducible covariants of the binary quintic for at this time he believed them to be infinite in number. His computational strategy continued to be gradualist and in the eighth memoir he produced two further covariants. His tally for the binary quintic was 17 irreducible covariants, each displayed in its full Cartesian form. When compared with the binary forms of order 3 and 4 the calculations for the quintic were especially difficult because of complexity in the inherent combinatorial problems. As Sylvester re-

marked wistfully to Cayley in a letter written in 1869: "But why should we expect to do this [solving a certain geometrical enumeration problem] . . . seeing how limited our powers of enumeration extend in the case of Invariants which have been so long the subject of study? (this consoling reflexion has only just occurred to me)" (I).

Nor would the symbolic method appear to circumvent these combinatorial problems. Indeed, one of Cayley's reservations about the symbolic method was that it failed to provide an *efficient* computational algorithm for finding the *irreducible* invariants and covariants. Cayley estimated that 429 transvection operations were necessary for the computation of the 23 irreducible invariants and covariants of the binary quintic. For higher order forms the transvection operation would be correspondingly poorer and Cayley remarked "the great excess of the number of derivatives over that of the covariants seems a reason why the derivatives ought not to be made a basis of the theory" [1878a; CP10, 378].

If calculation was regarded as a primary goal there is a sense in which both Cayley's method and the symbolic method were unsatisfactory when considered in isolation from each other. With the symbolic method there was no *a priori* procedure for deciding which of the generated invariants and covariants were actually irreducible. In the case of the binary cubic, for example, both $(uu)^2 = H$ and $(uH)^1 = \Phi$ are irreducible whereas $(u\Phi)^1 = H^2$ and is therefore reducible. Thus in the German method, if the irreducible invariants and covariants were desired then the reducible ones had to be identified and discarded. Failure to identify reducible ones would lead to an *overestimate* of the number of irreducible ones. Cayley had the opposite problem. He needed to find a putative invariant or covariant at the outset. Once this had been obtained it was relatively straightforward to investigate its reducibility using Cayley's Law. Lack of success in finding a putative invariant or covariant would lead him to *underestimate* the number of invariants and covariants. In this sense Cayley's method was synthetic in character while the German approach could be described as more analytical. The practitioners of invariant theory eventually viewed the two approaches as complementary, for if the German upper bound on the number of invariants and covariants coincided with the English lower bound then it was likely that a correct conclusion had been reached. Used in conjunction, the two approaches actually offered a procedure for cross checking calculations in a subject prone to numerical error.

The establishment of Gordan's theorem gave new stimulus to Cayley's work. While noting that its proof was difficult to understand he recognized it as a "theorem, the importance of which, in reference to the whole theory of forms, it is impossible to estimate too highly" [1871; CP7, 353]. Gordan's complete listing of the invariants and covariants for the binary quintic prompted him to produce the two missing covariants in his own list and thus he completed the tabulation of the 23 irreducible invariants and covariants of the binary quintic (4 invariants and 19 covariants) as Cartesian expressions [1871]. For him this was a milestone in the theory, but, seen in the light of his overt aim to "find all the derivatives [invariants] of any number of functions" [1846; CP1, 95], the cataloged results must have seemed meager. In 30 years Cayley's calculatory work for binary forms had progressed only as far as the binary form of order 5. And what is more, the binary

quintic still posed difficult questions concerning the *syzygies* (see glossary in Appendix). In the eighth memoir he had begun working on the problem of calculating these syzygies. A consequence of Cayley's gradualist method—advancing by increasing degree and order—was the appearance of new features and these continued to stimulate his research interest in the binary quintic. He found, for instance, that whereas between covariants of degree 5 there had been only one irreducible syzygy, for covariants of degree 6 there were no fewer than six. It is also notable that the problem of the syzygies was one that Cayley, with remarkable perception, had recognized 25 years earlier as one “which appears to present very great difficulties” [1846; CP1, 95].

Cayley's point of departure in invariant theory at the beginning of the 1870s was the Eulerian generating function. This could be used for enumeration but it could also be used for identifying invariants and covariants. Cayley refined the basic generating function by various transformations and the emphasis he placed on these transformations indicates that progress seemed to depend on finding an appropriate form of the generating function. It was by examining a so-called *real* generating function that Cayley finally listed the fundamental syzygies for the binary quintic.

Beyond the binary quintic mathematicians tackled higher order binary forms. The comment by E. T. Bell that the *American Journal of Mathematics* “began storing up sheaves of calculations against an imminent famine that has yet to arrive” is difficult to resist [Bell 1945, 428]. Claims were staked and long calculations embarked upon. Sylvester wanted the binary form of order 7, but writing to Cayley in 1878 added: “but [I] do not desire to preclude you from taking possession of the case of the 7c if you are particularly desirous to do so. But why not undertake the 9c? That is a gigantic labor which I would most willingly relinquish to you and which I know would yield certain new and interesting results” (9).

Carried out by hand, the calculations were formidable. Human computers were employed and paid through grants made available by the Royal Society of London and the British Association for the Advancement of Science. Cayley needed this help to finalize his listing of the irreducible covariants of the quintic, and for the period 1879–1882 these grants were used for completing covariant tables of binary quantics of orders 7, 8, 9, and 10 [Sylvester 1879; SP3, 311]. The impetus given to Cayley's investigations by Gordan's spectacular result proved short-lived. After the ninth memoir, which was published in 1871, the 10th memoir appeared in 1878, its eventual publication owing much to the success Sylvester was achieving at Johns Hopkins. In the 1880s Cayley and Sylvester turned more of their attention to *semi-invariants* (see glossary in Appendix) and further programs of a calculatory nature. To them, the wealth of detail was a sign of the richness of mathematics however tedious it seemed to later generations of mathematicians.

THE FUNDAMENTAL POSTULATE

In the urgency to find invariants and covariants, it was sometimes expedient to allow premises to rest on little more than intuition. One working assumption which Sylvester allowed to creep into the methods of calculation was the Funda-

mental Postulate. It was primarily introduced to safeguard his numerical process, for Sylvester was aware that without it the process might be incapable of revealing *all* invariants and covariants. The Fundamental Postulate effectively *limited* the number of syzygies which were supposed to exist. It is not clear whether its introduction by Sylvester was endorsed by Cayley, but it is reasonable to believe that he would also have accepted it as a working hypothesis. For Cayley, like his friend, possessed a pragmatic outlook.

Sylvester's numerical process, which he called *Tamisage* (see glossary in Appendix), involved careful scrutiny of the numerator and denominator of a generating function. By doing this he could identify the degree and order of each invariant and covariant. In assuming the truth of the Fundamental Postulate an "unconscious residue of countless past experiences"—Arthur Koestler's term [Koestler 1959, 335]—came into play. Sylvester assumed the postulate was true as he had never found an instance which contradicted it. Invoking it when necessary, he zealously assembled tables of invariants and covariants. As he told Cayley:

I think I may now announce with moral certainty that my method [of *Tamisage*] completely solves the problem of finding the *Grundformen* for binary forms and systems of binary forms . . . in all cases. . . . I ought to add that *anterior to all* verification this method *could not* give *superfluous* forms—but it is metaphysically conceivable that it might give *too few* *Grundformen*. The principle [Fundamental Postulate] I proceed upon is that in interpreting the generating function, we are not to assume the existence of more syzygetic relations than those which are necessary to make it consistent with itself and with the fact that every combination of Concomitants [invariants and covariants] is a Concomitant. (4)

The relationship between Sylvester's *Tamisage* process and the Fundamental Postulate is important. Sylvester's reasoning seems to be this: the actual number, say c , of linearly independent covariants of a specified degree and order could be calculated by Cayley's Law. If k ($k \geq c$) covariants were discovered through the *Tamisage* process then the Fundamental Postulate asserted that there were exactly $k - c$ syzygies between them. More syzygies would imply that the *Tamisage* process was incapable of revealing all c linearly independent covariants. The process had been found reliable in the cases of binary forms of low order and Sylvester felt secure in its application though he could not prove it. He found other evidence which did not contradict the postulate and he was quick to quote it "as another exemplification of the validity of that same very reasonable postulate" [1881; *SP*3, 509].

In 1882 Cayley visited Sylvester at Johns Hopkins University in America. In the same year, after Cayley had returned, a counterexample to the postulate was found by James Hammond [1882]. Cayley noted "the extreme importance of Mr. Hammond's result, as regards the entire subject of Covariants" [1883; *CP*11, 409]. Hammond's counterexample involved the binary form of order seven though Sylvester thought this was an exceptional case. Again it was his experience which led him to this belief and later mathematicians found that binary forms of *prime* degree were more difficult to penetrate than those of composite order. "Has it ever occurred to you," Sylvester wrote to Cayley in 1885, "to consider why my method in spite of a possible error in the result does as a matter of fact give *all* and

not only *some* of the seminvariants in all cases to which it has been applied, viz. 5c, 6c, 8c'' (17). A later result [Morley 1912, 47] showed that Sylvester's success in these specific cases happened to coincide with the cases for which the postulate *was* true. The "problem of the syzygies" bedeviled this calculatory work in invariant theory at this time and, as P. A. MacMahon later remarked, the method of generating functions "had become indeed a fruitful source of error" [MacMahon 1904, 9].

FINAL YEARS

A new direction emerged from the failure of the Fundamental Postulate. Sylvester was partly instrumental in this new impetus but the important theorem was due to a newcomer, the then relatively unknown P. A. MacMahon (1854–1929). MacMahon's theorem established a one-to-one correspondence between semi-invariants and the *nonunitary symmetric functions* (see glossary in Appendix) formed from the roots of binary forms of sufficiently high order. For this so-called infinite binary form

$$ax^n + \frac{b}{1!}x^{n-1}y + \frac{c}{2!}x^{n-2}y^2 + \frac{d}{3!}x^{n-3}y^3 + \dots$$

such symmetric functions as

$$\Sigma\alpha^2 = \frac{1}{a^2}(b^2 - ac),$$

$$\Sigma\alpha^2\beta^2 = \frac{1}{12a^2}(ae - 4bd + 3c^2)$$

indicate the correspondence. Sylvester had doubts as to the importance of the Correspondence Theorem but Cayley assured him that "the great use of MacMahon's theory is in the means which it affords for making out the whole theory of the syzygies. It is a question of double partitions" (16). Cayley was here referring to the fact that the tabulation of symmetric functions is equivalent to the arithmetical problem of partitioning an integer. Thus the Correspondence Theorem appears to offer a potential reduction of invariant theory to arithmetic. In character this observation is similar to Cayley's discovery of several years earlier that the study of abstract groups is potentially reducible to the study of permutation groups [1878b; CP10, 401–403].

According to MacMahon the reduction became a reality through the ensuing papers written by Cayley. In the 1880s MacMahon collaborated with Cayley on invariant theory and no doubt he had this work in mind when remarking that about the year 1885 Cayley was involved with a vast amount of purely numerical work [MacMahon 1896, 7]. Evidently the collaboration was close: "I am quite stopped by a question in Seminvariants partially solved by MacMahon," Cayley wrote Thomas Craig of Johns Hopkins in 1885, "but we are neither of us at present able

to make the next step; *if* I succeed in doing so, I should be rather inclined to undertake a treatise on the subject; but I do not at all see my way" (18). Involving algebraic forms expressed in the extensive Cartesian forms both Cayley's and MacMahon's method of work was extremely laborious. When dealing with the enumeration of irreducible semi-invariants of the binary quantic of infinite order MacMahon acknowledged that the "true method of procedure" in this work was via the German symbolic method [1910, 638].

The allure of Gordan's theorem continued to tempt both Cayley and Sylvester. According to MacMahon, Cayley's work "led him to desire a purely algebraic proof of Gordan's theorem concerning the finality of the covariants of quantics of finite order" [1896, 6]. Sylvester too was dissatisfied with Gordan's own proof which was, he claimed, "so long and complicated and so artificial a structure that it requires a very long study to master and there is not *one person* in Great Britain who *has* mastered it" (11). On his return to a chair at Oxford, Sylvester attempted to turn repeated failure into success. "In my off moments I have been thinking again of Gordan's theorem," he wrote Cayley in 1886, "and verily believe that I have found the proof . . . [Hammond] will check me if I am under any delusion as to the Gordanic business" (20). Two weeks later, a satisfactory proof still eluding him, he sent the cheerless note "I nourish the undying hope that . . . we shall be able to prove the finitude of the ground-forms of Invariants and Reciprocants by some simple process of reasoning" (21). As is well known, Hilbert provided several proofs of Gordan's theorem, and the first, concerned with binary forms, avoided the problem of actually constructing a basis. Cayley attempted to turn this proof into a constructive one by an argument based on semi-invariants. Though warned by Felix Klein that his argument was fallacious, Cayley insisted on publishing it. An appraisal of Cayley's proof is given in [Petersen 1890, 112].

In one of his last publications [1893a] Cayley tried to establish a criterion for the reducibility of covariants based on semi-invariants but MacMahon noted that the subject matter "bristled with difficulties and exceptional cases" [MacMahon 1896, 8]. Cayley's direct constructive approach, coupled with the inductive spirit of enquiry, was pitted against an exceptionally difficult problem—for which more sophisticated techniques were perhaps required.

There was also a brief encounter with the theory of reciprocants or differential invariants. "Am very glad you take an interest in my new functions—provisionally we may call them Reciprocants," Sylvester wrote Cayley from Oxford, "you will see that the whole of the game so to say of invariants has to be played out over again on a new field and subject to new laws but giving rise to a parallel theory of groundforms" (19).

Perhaps the prospect of beginning a new invariant theory was just too daunting for Cayley at this late stage, though he took sufficient interest in reciprocants to write a survey article [1893b]. Unlike the days of his first contributions to invariant theory, the range and quantity of mathematics now being produced were of vast proportions. Cayley, who believed that a man would automatically be aware of all new developments in his own subject, was unaware that Sylvester's pro-

posed new theory was properly subsumed under Sophus Lie's theory of transformation groups [Elliott 1898]. Perhaps more surprisingly, despite his knowledge of group theory, Cayley never attempted to merge its study with invariant theory (though in *modern terms* Cayley investigated the representations of the general linear group $GL(2, C)$ when he studied the binary quantic). The link with group theory was due to Lie, Klein, and other Continental mathematicians.

Cayley was from an earlier generation. He was concerned with such Continental developments in invariant theory as the German symbolic method. He saw this as an important development but not a replacement for the ordinary algebra of coordinates, and so he persevered in the task of tabulating invariants and covariants as Cartesian expressions. Indeed his principal mathematical interests, which included algebraic geometry, made it natural for him to treat invariant theory from the standpoint of coordinates. Seeing the actual algebraic forms paraded in this way was of prime importance for him: the forms, so expressed, did not have to be "unfolded" to be interpreted. Even when dealing with analytical geometry his natural way of thinking was through equations and not through the newly developed vector notation in which the reference to coordinates was suppressed. Cayley reaffirmed this central belief in coordinates when, delivering the presidential address at the British Association meeting of 1883, he declared: "Descartes' method of co-ordinates is a possession for ever."

CONCLUSION

Toward the end of his life, Cayley could look back on a half-century's involvement with invariant theory and perhaps with especial warmth on the first 20 years when he had been its formative influence. But in the 1860s the superiority of the German method was amply demonstrated by Gordan's incisive results. Still, the German symbolic method was not universally adopted; it was not until the new century that the method gained ground in England. Thus, according to the invariant theorist H. W. Turnbull, the publication in 1903 of Grace and Young's *Algebra of Invariants*, which was written in the symbolic notation, marked a "new era . . . for the teaching and progress of higher algebra" [Turnbull 1941, 767]. Mathematicians who were imbued with the spirit of the new higher algebra saw the arithmetical task of calculating the irreducible invariants and covariants as a sentimental quest and, for them, Cayley's work in invariant theory was happily forgotten. Others have seen the problems of invariant theory on which Cayley worked as having more lasting value and his papers continue to attract attention.

APPENDIX: GLOSSARY

Binary cubic. This is an algebraic form in two variables and order 3. It is expressed in its *Cartesian expression* by

$$u = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3$$

and in the German symbolic notation by

$$u = (\alpha_1 x_1 + \alpha_2 x_2)^3$$

or simply $u = \alpha_x^3$. The binary cubic possesses four *irreducible* algebraic forms with the invariant property. Expressed in the symbolic notation these are

$$u = \alpha_x^3 \quad (\text{degree 1, order 3}),$$

$$H = (\alpha\beta)^2 \alpha_x \beta_x \quad (\text{degree 2, order 2}),$$

$$\Phi = (\alpha\beta)^2 (\gamma\alpha) \beta_x \gamma_x^2 \quad (\text{degree 3, order 3}),$$

$$\nabla = (\alpha\beta)^2 (\alpha\gamma) (\beta\delta) (\gamma\delta)^2 \quad (\text{degree 4, order 0}).$$

Gordan's important lemma, with regard to the binary cubic, is illustrated by

$$H = (uu)^2,$$

$$\Phi = (uH)^1,$$

$$\nabla = (u\Phi)^3.$$

Cartesian expression. In this article the term "Cartesian expression" means a homogeneous algebraic form expressed in coefficients and variables. For example, the binary form of order n is:

$$a_0 x^n + a_1 \binom{n}{1} x^{n-1} y + a_2 \binom{n}{2} x^{n-2} y^2 + \cdots + a_n y^n.$$

Cayley's Law. The number of linearly independent covariants of *degree* θ and *order* s for a binary form of order n where $w = \frac{1}{2}(n\theta - s)$ is

$$P(0, 1, \dots, n)^\theta(\omega) - P(0, 1, \dots, n)^\theta(\omega - 1).$$

The symbol $P(0, 1, \dots, n)^\theta(\omega)$ is the number of partitions of ω into θ or fewer of 1, 2, 3, . . . , n [Cayley 1854a; CP2, 167].

Degree. The degree (grad) of a term in an algebraic form is the sum of exponential indices in the product of coefficients attached to that term (as distinct from the product of variables).

Irreducible. An algebraic form is irreducible if it cannot be expressed algebraically in terms of algebraic forms of lower degree and order.

Nonunitary symmetric function. This is a symmetric function

$$\Sigma \alpha^p \beta^q \dots \gamma^r$$

in which none of the indices is unity.

Order. The order (Ordnung) of a term in an algebraic form is the sum of the exponential indices in the variables attached to that term.

Semi-invariant (source). A semi-invariant of a binary form is the leading coefficient in the *Cartesian expression* of a covariant. The complete expression for a covariant can be deduced from the semi-invariant using Cayley's theorem [Cayley 1854a; CP2, 167]. The terminology is possibly chosen because a semi-invariant is only required to be annihilated by the *single* operator

$$\square = a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + \cdots + na_{n-1} \frac{\partial}{\partial a_n}.$$

Syzygy. This is a term used by Cayley and Sylvester for a linear relation between invariants and covariants. For the *binary cubic*

$$\Phi^2 - u^2\nabla + 4H^3 = 0$$

is a syzygy between the (composite) covariants

$$\Phi^2, \quad u^2\nabla, \quad H^3$$

of degree 6 and order 6.

Tamisation: This is the name given by Sylvester to a method of examining a generating function for identifying invariants and covariants and *syzygies* of a binary form. The terms of the numerator and denominator are “sifted” according to an algorithm. The technical details can be found in [Elliott 1913/1964, 175].

Transvection operation (Uebereinanderschabung). The k th transvectant of binary forms α_x^n and β_x^m is defined

$$(\alpha_x^n, \beta_x^m)^k = (\alpha, \beta)^k \alpha_x^{n-k} \beta_x^{m-k}.$$

Cayley’s hyperdeterminant derivative is

$$\overline{12} = \begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} \end{vmatrix}.$$

By applying $\overline{12}$ to $\alpha_x^n \beta_y^m$ and then identifying x and y ,

$$(\alpha_x^n, \beta_x^m)^1 = \frac{1}{nm} (\overline{12} \alpha_x^n \beta_y^m)_{x=y},$$

where $(\overline{12} \alpha_x^n \beta_y^m)_{x=y}$ is the *Jacobian* of α_x^n and β_x^m .

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NOTES

1. Brief explanations of some 19th-century terminology are given in the glossary in the Appendix.
2. It was natural for these pioneers to concentrate their attention on binary forms but invariants and covariants of single algebraic forms of more variables were also considered as were sets of these forms taken jointly.
3. Letters are numbered (r) in chronological order and referred to under the *Index of Documents* included in the References.

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The following abbreviations are used in the list of manuscript sources: AC, Arthur Cayley; TC, Thomas Craig; FK, Felix Klein; WS, William Spottiswoode; JJS, James Joseph Sylvester; JT, John Tyndall; N.S.U.G., Niedersächsische Staats-und Universitätsbibliothek Göttingen; J.H.U., Johns Hopkins University; St. J., St. John's College, Cambridge; R.I., Royal Institute of Great Britain.

Number	Date		Author/Recipient	Location
(1)	21 June	1869	JJS/AC	St.J.
(2)	19 November	1876	JJS/WS	St.J.
(3)	25 November	1876	JJS/WS	St.J.
(4)	23 April	1877	JJS/AC	St.J.
(5)	9 May	1877	JJS/AC	St.J.
(6)	6 November	1877	JJS/AC	St.J.
(7)	21 December	1877	JJS/AC	St.J.
(8)	7 February	1878	AC/JJS	St.J.
(9)	15 July	1878	JJS/AC	St.J.
(10)	11 January	1879	JJS/AC	St.J.
(11)	14 September	1882	JJS/JT	R.I.
			[Tyndall Corr. Vol. 4, 1519]	
(12)	6 October	1882	JJS/AC	St.J.
(13)	25 December	1882	JJS/AC	St.J.
(14)	12 February	1883	AC/JJS	St.J.
(15)	9 August	1883	JJS/AC	St.J.
(16)	11 August	1883	AC/JJS	St.J.
(17)	11 May	1885	JJS/AC	St.J.
(18)	12 August	1885	AC/TC	J.H.U.
(19)	24 October	1885	JJS/AC	St.J.
(20)	1 February	1886	JJS/AC	St.J.
(21)	18 February	1886	JJS/AC	St.J.
(22)	24 January	1889	AC/FK	N.S.U.G.
			[Cod. Ms. F. Klein 8:383]	
(23)	22 February	1889	AC/FK	N.S.U.G.
			[Cod. Ms. F. Klein 8:384]	

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